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Thermal problem of curvilinear cracks in bonded dissimilar materials with a point heat source

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Abstract—Boundary value problems for a circular-arc crack embedded in dissimilar materials under the application of a point heat source are formulated and solved in closed form. Based on the Hilbert problem formulation and a special technique of analytic continuation, exact solutions of the temperature and temperature gradient are obtained in an explicit form. It is found that the temperature gradients or heat fluxes near the crack tips of a curved crack possess the characteristic inverse square-root singularity in terms of the radial distance away from the crack tip which is the same as those obtained for a straight crack between dissimilar materials. Due to this singular behavior, the heat flux intensity factor is introduced to measure the thermal energy intensification cumulated in the vicinity of the crack tip. Numerical results for the temperature and heat flux intensity factor are provided in graphic form. It is shown that the thermal system having a smaller crack length would make the heat flux intensity factor lower. Consequently, the thermal energy intensification is diminished.

INTRODUCTION

THERMAL problems concerning heat sources are frequently encountered in many sectors of modern technology. Such problems may occur, for example, in a nuclear reactor core and chemical process equipment. There arose the problem of finding temperature and heat flux distribution in dissimilar materials containing imperfections in the form of interface cracks under the application of heat sources. Problems of this kind present considerable mathematical difficulty. In particular, when the geometry of interest is multiply connected, the closed form solution is difficult to achieve. The singularity of $1/\sqrt{\rho}$ of the temperature gradient near the crack tip for an infinite cracked plate under the uniform heat flow was first derived by Sih [1]. The value of ρ here stands for the radial distance measured from the crack tip. Using the Hilbert problem formulation [2] and a special technique of analytic continuation, Chao and Chang [3] gave a simple and compact version of the general solution for the thermal interface crack problem in dissimilar anisotropic media. They found that the temperature gradients or heat fluxes near the crack tip always possess the characteristic inverse square-root singularity regardless of material anisotropy as the heat conductivity coefficients obey the reciprocal relation, $k_{ij} = k_{ji}$, $(i \neq j)$.

In this paper, the emphasis will be placed on the determination of the temperature function due to a point heat source embedded in dissimilar materials with a circular-arc crack. This problem is solved by reduction to a boundary value problem of complex variable theory in conjunction with a special technique of analytic continuation. Exact formulae for the temperatures and temperature gradients are obtained in closed form and related numerical values of temperature and heat flux intensity factor [4] are provided in graphic form. For a limiting case of a crack with sufficiently small arc length, the heat flux intensity factor is found to be linearly proportional to the square root of the crack length which is very similar to the result pertaining to a straight crack in the isotropic, homogeneous medium given by Chao and Chang [4].

PROBLEM STATEMENT

Consider two homogeneous isotropic materials, one occupies the region S^+, interior to the unit circle $r = 1$, while the other occupies the infinite region, S^-, exterior to the unit circle as shown in Fig. 1. The heat conductivity of the material in S^+ is specified by the constant k_1 and that of the material in S^- by k_2. If the bond between the two materials on the unit circle is imperfect, it can be represented as the sum of L and L^*, where L and L^* stand for the segments of circular-arc crack and circular-arc bond, respectively. The heat flux due to a point heat source is disturbed by the presence of an insulated circular-arc lying in the unit circle with the center placed at the location of a point heat source. The boundary conditions along the crack surface can be stated as

$$q_{r1}^{+} = 0 \quad \text{on } L \tag{1}$$

$$q_{r2}^{-} = 0 \quad \text{on } L \tag{2}$$

where q_{rj} ($j = 1$ for S^+ and $j = 2$ for S^-) denote the radial heat fluxes and the supercripts $+$ and $-$ are

NOMENCLATURE

a, b	ends of crack
A_i	constant coefficients ($i = 1, 2, \ldots, n$)
B_i	constant coefficients ($i = 1, 2, \ldots, n$)
D_i	constant coefficients ($i = 1, 2$)
E_i	constant coefficients ($i = 1, 2$)
F_0	resultant heat flux exerted on crack surface
h	net heat flux
$h(\theta)$	heat flux prescribed on the crack surface
H	heat flux intensity factor
k_i	conductivity coefficients for isotropic material ($i = 1, 2$)
k_{ij}	conductivity coefficients for anisotropic material ($i, j = 1, 2$)
L	segment of circular-arc crack
L^*	segment of circular-arc bond
$P(z)$	complex polynomial
q_0	constant rate of point heat source
q_{rj}	radial heat flux ($j = 1, 2$)
r	radius of the circular-arc crack
$Re\{\ \}$	real part of a complex function
T	temperature
x, y	rectangular coordinate axes
z	complex coordinate, $x+iy$.

Greek symbols

α	half angle of crack
Γ_0	applied temperature gradient at infinity
δ_i	coefficients of the polynomial $P(z)$
θ	polar angle of the complex plane
$\Theta_j(z)$	temperature gradient functions for flawed plate ($j = 1, 2$)
ρ	radial distance from the crack tip
ϕ	polar angle at the crack tip
$\phi_{uj}(z)$	temperature functions for unflawed plate ($j = 1, 2$)
$\phi_{fj}(z)$	temperature functions for flawed plate ($j = 1, 2$)
$\Phi_{uj}(z)$	temperature gradient functions for unflawed plate, $\phi'_{uj}(z)$ ($j = 1, 2$)
$\Phi_{fj}(z)$	temperature gradient functions for flawed plate, $\phi'_{fj}(z)$ ($j = 1, 2$).

used to denote the boundary values of the physical quantities as they are approached from S^+ and S^-, respectively.

It is convenient to represent the solution as the sum of the heat flux due to a point heat source in an unflawed plate (Fig. 2(a)) and a corrective solution (Fig. 2(b)), for which the boundary conditions are

$$q_{r1}^+ = -h(\theta) \quad \text{on } L \tag{3}$$

$$q_{r2} = -h(\theta) \quad \text{on } L \tag{4}$$

where the unknown function $h(\theta)$ will be determined from the solution corresponding to the problem of a point heat source in an unflawed plate.

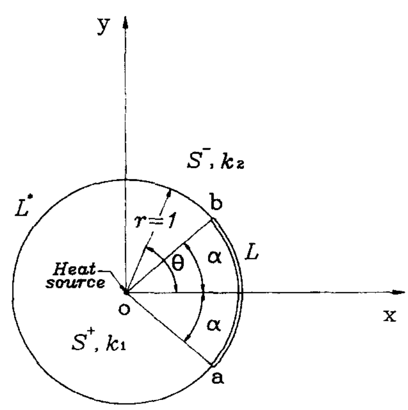

FIG. 1. A single circular-arc crack lying along the interface of two dissimilar materials with a point heat source.

TEMPERATURE FIELD

Let the point heat source with a constant rate q_0 per unit time be located at the origin of the complex plane $z = x+iy$. For the two-dimensional steady state heat conduction problem, the temperature field for the unflawed plate satisfies the Laplace equation

$$\nabla^2 T = 0. \tag{5}$$

Both the heat fluxes and temperatures are continuous along the interface of bonded dissimilar materials, they are

$$-k_1 \frac{\partial T_1}{\partial r} = -k_2 \frac{\partial T_2}{\partial r} = \frac{q_0}{2\pi} \quad \text{on } L+L^* \tag{6}$$

$$T_1 = T_2 \quad \text{on } L+L^*. \tag{7}$$

The solutions associated with equation (5) with boundary conditions from equations (6) and (7) are

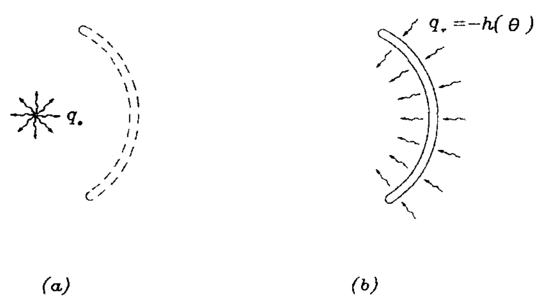

FIG. 2. Superposition of heat flux in an unflawed plate and a corrective problem.

$$T_j = -\frac{q_0}{2\pi k_j}\ln r$$

$$j = \begin{cases} 1, & z\in S^+ \\ 2, & z\in S^- \end{cases}. \tag{8}$$

In view of equation (8), the temperature is set to be zero along the interface without loss of generality. The temperature $T_j(x,y)$ can also be related to the analytic function $\phi_{uj}(z)$ in the form

$$T_j(x,y) = Re\,[\phi_{uj}(z)] \tag{9}$$

and

$$\Phi_{uj}(z) = \phi'_{uj}(z) = -\frac{q_0}{2\pi k_j z} \tag{10}$$

where the subindex u denotes the unflawed plate, and Re stands for the real part of the complex function. Having the solution associated with the unflawed plate as indicated in equation (8), the unknown function $h(\theta)$ in equations (3) and (4) can then be obtained as

$$h(\theta) = \frac{q_0}{2\pi}. \tag{11}$$

In order to specify the boundary conditions on the crack surface of the flawed plate, the radial heat fluxes q_{rj} and circumferential heat fluxes $q_{\theta j}$ are introduced in terms of the temperature gradient $\Phi_{fj}(z)$ by the following equation

$$q_{rj}+\mathrm{i}q_{\theta j} = -k_j\overline{\Phi_{fj}(z)}\,\mathrm{e}^{-\mathrm{i}\theta} \tag{12}$$

where the subindex f denotes the flawed plate and the overbar stands for the complex conjugate.

Let the center of the unit circle be placed at the origin of the complex plane, $z = x+\mathrm{i}y$ and $t = \exp(\mathrm{i}\theta)$ be those points of z on $|z| = 1$. For this problem, the radial heat flux q_r will be specified on L while the continuity of the radial heat flux and temperature are required on L^*, i.e.

$$q_{r1}^+ = q_r^+ = -\frac{q_0}{2\pi} \quad \text{on } L \tag{13}$$

$$q_{r2}^- = q_r^- = -\frac{q_0}{2\pi} \quad \text{on } L \tag{14}$$

and

$$q_{r1} = q_{r2} \quad \text{on } L^* \tag{15}$$

$$T_1 = T_2 \quad \text{on } L^*. \tag{16}$$

For problems involving arc of discontinuity, it is convenient to further introduce the functions

$$\Theta_j(z) = \frac{1}{z^2}\Phi_{fj}\left(\frac{1}{z}\right), \quad j = 1,2. \tag{17}$$

Making use of equations (12) and (17), equations (13)–(16) can now be expressed in terms of $\Phi_{fj}(z)$, $\Theta_j(z)$ as

$$[\Phi_{f1}^+(t)+\Theta_1^-(t)] = \frac{-2q_r^+\,\mathrm{e}^{-\mathrm{i}\theta}}{k_1} \quad \text{on } L \tag{18}$$

$$[\Phi_{f2}^-(t)+\Theta_2^+(t)] = \frac{-2q_r^-\,\mathrm{e}^{-\mathrm{i}\theta}}{k_2} \quad \text{on } L \tag{19}$$

and

$$k_1[\Phi_{f1}(t)+\Theta_1(t)] = k_2[\Phi_{f2}(t)+\Theta_2(t)] \quad \text{on } L^* \tag{20}$$

$$[\Phi_{f1}(t)-\Theta_1(t)] = [\Phi_{f2}(t)-\Theta_2(t)] \quad \text{on } L^*. \tag{21}$$

It should be noted that equation (21) requires only the derivatives

$$\frac{\partial T_1}{\partial\theta} = \frac{\partial T_2}{\partial\theta}$$

be continuous across L^* instead of the temperature T itself as indicated in equation (16). Hence, a complete solution to the bi-material crack problem has been reduced to the evaluation of four complex functions $\Phi_{fj}(z)$, $\Theta_j(z)$, $(j = 1,2)$, which must satisfy the conditions on L and L^* as given by equations (18)–(21).

Starting from the assumptions that the radial heat flux and temperature are continuous over the bonded segment of the circle $|z| = 1$, equations (20) and (21) may be regarded as the conditions of analytic continuation of $\Phi_{fj}(z)$, $\Theta_j(z)$ from S^+ to S^- across L^*. Now, $\Phi_{f1}(t)$ and $\Theta_1(t)$ in equations (20) and (21) may be solved explicitly in terms of $\Phi_{f2}(t)$ and $\Theta_2(t)$, and the resulting expressions are valid everywhere in the z-plane as

$$\Phi_{f1}(z) = \frac{k_2+k_1}{2k_1}\Phi_{f2}(z)+\frac{k_2-k_1}{2k_1}\Theta_2(z) \tag{22}$$

$$\Theta_1(z) = \frac{k_2-k_1}{2k_1}\Phi_{f2}(z)+\frac{k_2+k_1}{2k_1}\Theta_2(z). \tag{23}$$

Inserting these equations into the boundary conditions, equations (18) and (19), and solving them simultaneously yields

$$[\Phi_{f2}(t)+\Theta_2(t)]^+ + [\Phi_{f2}(t)+\Theta_2(t)]^- = 2f(t) \tag{24}$$

$$[\Phi_{f2}(t)-\Theta_2(t)]^+ - [\Phi_{f2}(t)-\Theta_2(t)]^- = 2g(t) \tag{25}$$

where $f(t)$ and $g(t)$ are related to the heat flux on L by

$$f(t) = \frac{-2k_1\,\mathrm{e}^{-\mathrm{i}\theta}}{k_1+k_2}\left(\frac{q_r^+}{k_1}+\frac{q_r^-}{k_2}\right) \tag{26}$$

$$g(t) = \frac{-2\,\mathrm{e}^{-\mathrm{i}\theta}}{k_1+k_2}(q_r^+ - q_r^-). \tag{27}$$

Knowing that equation (25) is a Plemelj equation for the function $\Phi_{f2}(z)-\Theta_2(z)$ and it has the solution [2]

$$\Phi_{f2}(z)-\Theta_2(z) = \frac{1}{\pi\mathrm{i}}\int_L \frac{g(t)}{t-z}\,\mathrm{d}t+E_0+\frac{E_1}{z}+\frac{E_2}{z^2}. \tag{28}$$

Furthermore, the non-homogeneous Hilbert equation, equation (24), gives [2]

$$\Phi_{f2}(z)+\Theta_2(z) = \frac{X_0(z)}{\pi i}\int_L \frac{f(t)}{X_0^+(t)(t-z)}\,dt + X_0(z)\left[P(z)+\frac{D_1}{z}+\frac{D_2}{z^2}\right] \quad (29)$$

where the Plemelj function

$$X_0(z) = (z-a)^{-1/2}(z-b)^{-1/2} \quad (30)$$

which provides the necessary branch cut and is selected such that

$$\lim_{z\to\infty}[zX_0(z)] = 1.$$

Note that the crack-tip heat fluxes possess the characteristic inverse square-root singularity as indicated in equation (30) which is not affected by the discontinuous conductivity jumps across the material interface. In the present problem, the ends of the crack L are located at $a = \exp(-i\alpha)$ and $b = \exp(i\alpha)$ and the Plemelj function in equation (30) yields

$$X_0(z) = \frac{1}{\sqrt{(z^2-2z\cos\alpha+1)}}. \quad (31)$$

The polynomial $P(z)$ in equation (29) is expressed as

$$P(z) = \delta_0 z+\delta_1. \quad (32)$$

By means of equations (28), (29) and (22), the general solutions for the functions $\Phi_{f1}(z)$, $\Phi_{f2}(z)$ may be expressed as follows

$$\Phi_{f1}(z) = \frac{1}{2\pi i}\int_L \frac{g(t)}{t-z}\,dt + \frac{k_2 X_0(z)}{2k_1\pi i}\int_L \frac{f(t)}{X_0^+(t)(t-z)}\,dt + \frac{E_0}{2}+\frac{E_1}{2z}+\frac{E_2}{2z^2}+\frac{k_2X_0(z)}{2k_1}\left[\delta_0 z+\delta_1+\frac{D_1}{z}+\frac{D_2}{z^2}\right] \quad (33)$$

$$\Phi_{f2}(z) = \frac{1}{2\pi i}\int_L \frac{g(t)}{t-z}\,dt + \frac{X_0(z)}{2\pi i}\int_L \frac{f(t)}{X_0^+(t)(t-z)}\,dt + \frac{E_0}{2}+\frac{E_1}{2z}+\frac{E_2}{2z^2}+\frac{X_0(z)}{2}\left[\delta_0 z+\delta_1+\frac{D_1}{z}+\frac{D_2}{z^2}\right]. \quad (34)$$

Using the boundary conditions from equations (13) and (14), the functions $f(t)$ and $g(t)$ appearing in equations (26) and (27), respectively, now reduce to

$$f(t) = \frac{q_0}{\pi k_2 t} \quad (35)$$

and

$$g(t) = 0. \quad (36)$$

With the aid of equations (35), (36) and evaluating the Cauchy integral, equations (33), (34) can be put in the form

$$\Phi_{f1}(z) = \frac{q_0}{2\pi k_1}\left[\frac{1}{z}+\frac{X_0(z)}{z}-X_0(z)\right] + \frac{E_0}{2}+\frac{E_1}{2z}+\frac{E_2}{2z^2}+\frac{k_2}{2k_1}X_0(z)\left[\delta_0 z+\delta_1+\frac{D_1}{z}+\frac{D_2}{z^2}\right] \quad (37)$$

$$\Phi_{f2}(z) = \frac{q_0}{2\pi k_2}\left[\frac{1}{z}+\frac{X_0(z)}{z}-X_0(z)\right] + \frac{E_0}{2}+\frac{E_1}{2z}+\frac{E_2}{2z^2}+\frac{X_0(z)}{2}\left[\delta_0 z+\delta_1+\frac{D_1}{z}+\frac{D_2}{z^2}\right]. \quad (38)$$

The determination of the unknown constants appearing in equations (37) and (38) calls for a knowledge of the behavior of the complex functions for small and large values of $|z|$. First of all, since $\Phi_{f1}(z)$ is holomorphic in S^+, it must take the form

$$\Phi_{f1}(z) = A_0+A_1z+A_2z^2+\cdots \quad \text{for } |z|<1. \quad (39)$$

Substituting equations (39) into (17), $\Theta_1(z)$ is found to be holomorphic in S^-. Therefore

$$\Theta_1(z) = \frac{\bar{A}_0}{z^2}+\frac{\bar{A}_1}{z^3}+\cdots \quad \text{for } |z|>1. \quad (40)$$

Similarly, as $\Phi_{f2}(z)$ is holomorphic in S^-, it must take the form

$$\Phi_{f2}(z) = \Gamma_0+\frac{F_0}{z}+O\left(\frac{1}{z^2}\right)+\cdots \quad \text{for } |z|>1 \quad (41)$$

where F_0 denotes the resultant heat flux exerted on L and Γ_0 is the applied temperature gradient at infinity, i.e.

$$\Gamma_0 = \left(\frac{\partial T}{\partial x}\right)\bigg|_{|z|\to\infty} - i\left(\frac{\partial T}{\partial y}\right)\bigg|_{|z|\to\infty}. \quad (42)$$

Substituting equation (41) into (17), $\Theta_2(z)$ is found to be holomorphic in S^+. Therefore

$$\Theta_2(z) = \frac{\bar{\Gamma}_0}{z^2}+\frac{\bar{F}_0}{z}+\text{a holomorphic function} \quad \text{for } |z|<1 \quad (43)$$

which is holomorphic in S^+ with the exception of the point $z = 0$. In the same way, the definition of the function $\Phi_{f2}(z)$ may be extended into the region S^+ by substituting equations (39), (43) into equation (22)

$$\Phi_{f2}(z) = \frac{B_2}{z^2}+\frac{B_1}{z}+\text{a holomorphic function} \quad \text{for } |z|<1 \quad (44)$$

where

$$B_1 = \frac{k_1-k_2}{k_1+k_2}\bar{F}_0,\quad B_2 = \frac{k_1-k_2}{k_1+k_2}\bar{\Gamma}_0. \quad (45)$$

Moreover, the function $\Theta_2(z)$ in the region S^- can be found by substituting equations (40) and (41) into equation (23). It yields

$$\Theta_2(z) = \bar{B}_2 + \frac{\bar{B}_1}{z} + O\left(\frac{1}{z^2}\right) + \cdots \quad \text{for } |z| > 1. \tag{46}$$

The constants E_1 and E_2 in equation (28) may be associated with the coefficients of the series given by equations (43) and (44) as

$$E_1 = B_1 - \bar{F}_0 = \frac{-2k_2}{k_1+k_2}\bar{F}_0 \tag{47}$$

$$E_2 = B_2 - \bar{\Gamma}_0 = \frac{-2k_2}{k_1+k_2}\bar{\Gamma}_0. \tag{48}$$

Similarly, the constant E_0 in equation (28) can also be found from the behavior of the complex function $\Phi_{r2}(z) - \Theta_2(z)$ for large values of $|z|$, it gives

$$E_0 = \Gamma_0 - \bar{B}_2 = \frac{2k_2}{k_1+k_2}\Gamma_0. \tag{49}$$

In the same manner, the constants δ_0 and δ_1 in equation (32) may be determined from the behavior of the complex function $\Phi_{r2}(z) + \Theta_2(z)$ at infinity by expanding the Plemelj function $X_0(z)$ for large value of $|z|$. It gives

$$\delta_0 = \Gamma_0 + \bar{B}_2 = \frac{2k_1}{k_1+k_2}\Gamma_0 \tag{50}$$

$$\delta_1 = \bar{B}_1 + F_0 - \delta_0 \cos\alpha = \frac{2k_1}{k_1+k_2}(F_0 - \Gamma_0\cos\alpha). \tag{51}$$

In addition, the constants D_1 and D_2 in equation (29) can also be evaluated from

$$X_0(z)\left(\frac{D_1}{z} + \frac{D_2}{z^2}\right) = \frac{B_2 + \bar{\Gamma}_0}{z^2} + \frac{B_1 + \bar{F}_0}{z} \tag{52}$$

near the point $|z| = 0$. By expanding the Plemelj function $X_0(z)$ for small value of $|z|$, and using equation (52), we have

$$D_2 = -(B_2 + \bar{\Gamma}_0) = -\frac{2k_1}{k_1+k_2}\bar{\Gamma}_0 \tag{53}$$

$$D_1 = \frac{-2k_1}{k_1+k_2}(\bar{F}_0 - \bar{\Gamma}_0\cos\alpha). \tag{54}$$

The remaining unknown, F_0, is to be found from the condition that the temperature must be single-valued, i.e. the temperature must revert to its original value as the point z describes a contour around a given segment, say L. Applying equation (21), such a requirement is equivalent to

$$\int_L [\Phi_{r1}^+(t) - \Theta_1^-(t)]\,dt - \int_L [\Phi_{r2}^-(t) - \Theta_2^+(t)]\,dt = 0. \tag{55}$$

For the purpose of computation in subsequent work, equations (22) and (23) may be used to put equation (55) in the form

$$\int_L \{[\Phi_{r2}(t) + \Theta_2(t)]^+ - [\Phi_{r2}(t) + \Theta_2(t)]^-\}\,dt = 0. \tag{56}$$

The result is

$$F_0 = 0. \tag{57}$$

Now, all the constants appearing in equations (33) and (34) have been obtained and the temperature gradient functions $\Phi_{r1}(z)$ and $\Phi_{r2}(z)$ become

$$\Phi_{r1}(z) = \frac{q_0}{2\pi k_1}\left[\frac{1}{z} + \frac{X_0(z)}{z} - X_0(z)\right] + \frac{k_2}{k_1+k_2}\left\{\Gamma_0 - \frac{\bar{\Gamma}_0}{z^2} + X_0(z)\left[\Gamma_0 z - \Gamma_0\cos\alpha + \frac{\bar{\Gamma}_0\cos\alpha}{z} - \frac{\bar{\Gamma}_0}{z^2}\right]\right\} \tag{58}$$

$$\Phi_{r2}(z) = \frac{q_0}{2\pi k_2}\left[\frac{1}{z} + \frac{X_0(z)}{z} - X_0(z)\right] + \frac{k_2}{k_1+k_2}\left\{\Gamma_0 - \frac{\bar{\Gamma}_0}{z^2} + \frac{k_1}{k_2}X_0(z)\left[\Gamma_0 z - \Gamma_0\cos\alpha + \frac{\bar{\Gamma}_0\cos\alpha}{z} - \frac{\bar{\Gamma}_0}{z^2}\right]\right\}. \tag{59}$$

If the applied temperature gradient at infinity vanishes, i.e. $\Gamma_0 = 0$, equations (58) and (59) can further reduce to

$$\Phi_{r1}(z) = \frac{q_0}{2\pi k_1}\left[\frac{1}{z} + \frac{X_0(z)}{z} - X_0(z)\right] \tag{60}$$

$$\Theta_{r2}(z) = \frac{q_0}{2\pi k_2}\left[\frac{1}{z} + \frac{X_0(z)}{z} - X_0(z)\right]. \tag{61}$$

The solutions associated with the boundary conditions (1) and (2) can then be obtained by applying the method of superposition as

$$\Phi_1(z) = \Phi_{u1}(z) + \Phi_{r1}(z) = \frac{q_0}{2\pi k_1}\left[\frac{X_0(z)}{z} - X_0(z)\right] \tag{62}$$

$$\Phi_2(z) = \Phi_{u2}(z) + \Phi_{r2}(z) = \frac{q_0}{2\pi k_2}\left[\frac{X_0(z)}{z} - X_0(z)\right]. \tag{63}$$

Integrating equations (62) and (63) with respect to z, the temperature functions $\phi_1(z)$ and $\phi_2(z)$ can be obtained as

$$\phi_1(z) = \frac{-q_0}{2\pi k_1}\left[\ln(2\sqrt{(1-2z\cos\alpha+z^2)} + 2z - 2\cos\alpha) + \ln\left(\frac{2\sqrt{(1-2z\cos\alpha+z^2)} - 2z\cos\alpha + 2}{z}\right)\right] \tag{64}$$

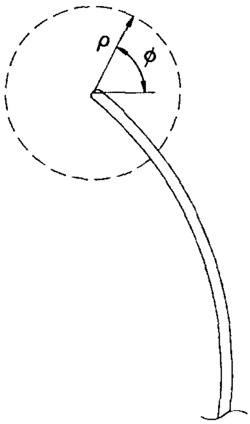

FIG. 3. Polar coordinate at the crack tip.

$$\phi_2(z) = \frac{-q_0}{2\pi k_2}\Bigg[\ln(2\sqrt{(1-2z\cos\alpha+z^2)}+2z-2\cos\alpha)$$

$$+\ln\left(\frac{2\sqrt{(1-2z\cos\alpha+z^2)}-2z\cos\alpha+2}{z}\right)\Bigg] \quad (65)$$

for $\alpha \neq 0^\circ$, and

$$\phi_1(z) = \frac{-q_0}{2\pi k_1}\left[\ln(z-1)-\ln\left(\frac{z-1}{z}\right)\right] \quad (66)$$

$$\phi_2(z) = \frac{-q_0}{2\pi k_2}\left[\ln(z-1)-\ln\left(\frac{z-1}{z}\right)\right] \quad (67)$$

for $\alpha = 0^\circ$.

In view of equations (62)–(67), it is noted that the solutions of temperature or temperature gradient pertaining to the inner and outer media of dissimilar materials are dependent on their own material properties.

HEAT FLUX SINGULARITY

In order to examine the local behavior of the temperature gradients in the vicinity of the crack tips, the polar coordinate system (ρ, ϕ) centered at the crack tip is considered in the present work (Fig. 3). Now, we restrict attention to a small region surrounding the crack tip $z = a$, equations (62) and (63) take the approximate forms

$$\Phi_1(z) = \frac{q_0}{2\pi k_1}\frac{F(\alpha,\phi)}{\sqrt{(2\rho)}}+O(1)$$

$$\text{for } \pi/2-\alpha < \phi < 3\pi/2-\alpha \quad (68)$$

$$\Phi_2(z) = \frac{q_0}{2\pi k_2}\frac{F(\alpha,\phi)}{\sqrt{(2\rho)}}+O(1)$$

$$\text{for } -\pi/2-\alpha < \phi < \pi/2-\alpha \quad (69)$$

where

$$F(\alpha,\phi) = \frac{\exp\left[-\left(\frac{\phi}{2}-\frac{\pi}{4}\right)i\right]}{\sqrt{(\sin\alpha)}}[e^{i\alpha}-1].$$

Similarly, the local temperature gradients near the crack tip $z = b$ are

$$\Phi_1(z) = \frac{q_0}{2\pi k_1}\frac{G(\alpha,\phi)}{\sqrt{(2\rho)}}+O(1)$$

$$\text{for } \pi/2+\alpha < \phi < 3\pi/2+\alpha \quad (70)$$

$$\Phi_2(z) = \frac{q_0}{2\pi k_2}\frac{G(\alpha,\phi)}{\sqrt{(2\rho)}}+O(1)$$

$$\text{for } \alpha-\pi/2 < \phi < \alpha+\pi/2 \quad (71)$$

where

$$G(\alpha,\phi) = \frac{\exp\left[-\left(\frac{\phi}{2}+\frac{\pi}{4}\right)i\right]}{\sqrt{(\sin\alpha)}}[e^{-i\alpha}-1].$$

It is seen that the local temperature gradients possess the characteristic inverse square-root singularity in terms of the radial distance, ρ, from the tips of the crack. Due to this singular behavior, the heat flux intensity factor is then introduced to quantify the thermal energy intensification in the vicinity of the crack tip which is defined as [4]

$$H = \lim_{\rho\to 0}\sqrt{(2\rho)}h \quad (72)$$

where h is the net heat flux given by

$$h = \sqrt{((q_x)_j^2+(q_y)_j^2)}$$

$$= k_j\sqrt{(\Phi_j(z)\overline{\Phi_j(z)})} \quad (j = 1, 2). \quad (73)$$

Substituting equations (68) and (69) or (70) and (71) into equation (73) and using equation (72), the factor H at crack tip $z = a$ or $z = b$ is found to be

$$H = \frac{q_0}{\sqrt{2\pi}}\sqrt{\left(\tan\frac{\alpha}{2}\right)}. \quad (74)$$

It is interesting to see that the heat flux intensity factor depends only on the crack length and the strength of a point heat source. If we let the crack angle α be sufficiently small, the factor H becomes

$$H = \frac{q_0\sqrt{\alpha}}{2\pi}. \quad (75)$$

It is realized that the thermal energy intensification disappears ($H = 0$) for the problem with a perfect bond by letting $\alpha = 0^\circ$.

NUMERICAL RESULTS

The general solutions to the thermal problem of curvilinear crack with a point heat source have been presented in equations (62)–(65). Some numerical

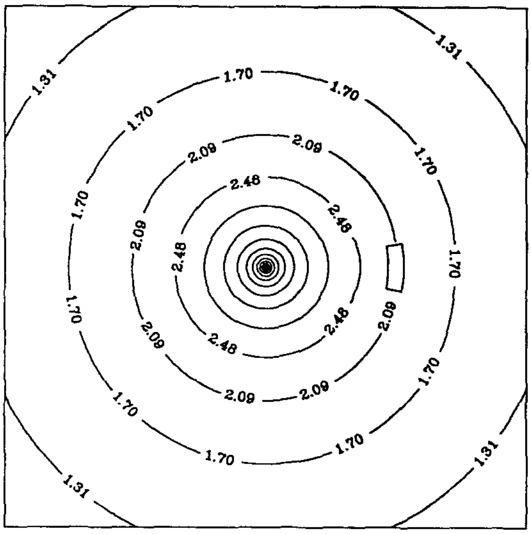

FIG. 4. Isothermal contours for $k_1/k_2 = 1$, $\alpha = 10°$.

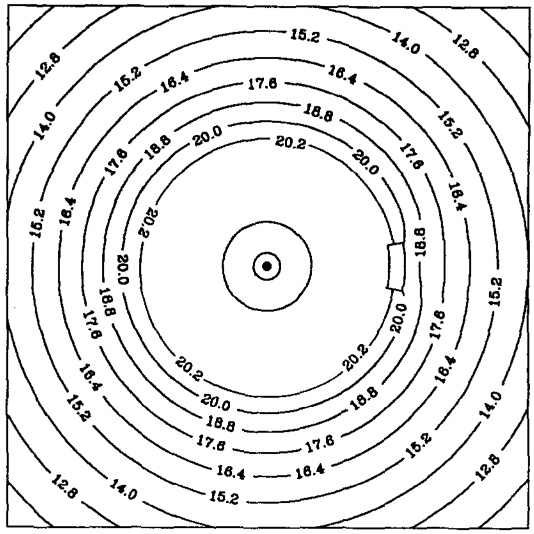

FIG. 6. Isothermal contours for $k_1/k_2 = 10$, $\alpha = 10°$.

results are given to illustrate the full field temperature distribution as well as the heat flux intensity factor.

ISOTHERMAL CONTOUR

A detailed understanding of the temperature distribution is useful to examine the global instability of the thermal system. Referring to Fig. 1, the heat flow due to the point heat source with a constant rate q_0 per unit time is obstructed by the presence of an insulated circular-arc crack with an angle α ranging from 0° to 180°. All the isothermal contours possess the unit $q_0/\pi k_1$ which would reflect the combined effects of geometric singularity around tips of the crack and thermal properties of the solid media. Figures 4–6 indicate the effect on the temperature contours across the interface between dissimilar materials by varying the heat conductivity ratio, k_1/k_2, for a relatively small crack $\alpha = 10°$. It is shown that the presence of the crack has little influence on the temperature contours as displayed in Fig. 4 for the homogeneous material ($k_1/k_2 = 1$). Similar observations can be made for the nonhomogeneous materials $k_1/k_2 = 0.1$ and $k_1/k_2 = 10$ as displayed in Figs. 5 and 6, respectively. As the heat conductivity of the outer material dominates over that of the inner material ($k_1/k_2 < 1$), the gradient of the constant temperature contours for the inner material is larger than that of the outer material in order to maintain the constant heat flow across the interface. On the contrary, the gradient of the constant temperature contours for the outer material is higher than that of the inner material

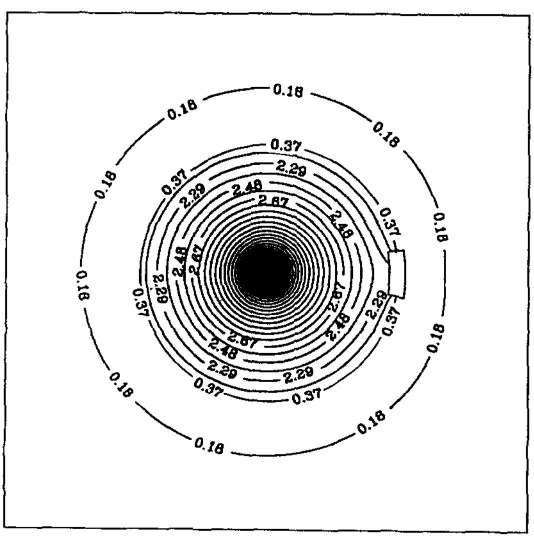

FIG. 5. Isothermal contours for $k_1/k_2 = 0.1$, $\alpha = 10°$.

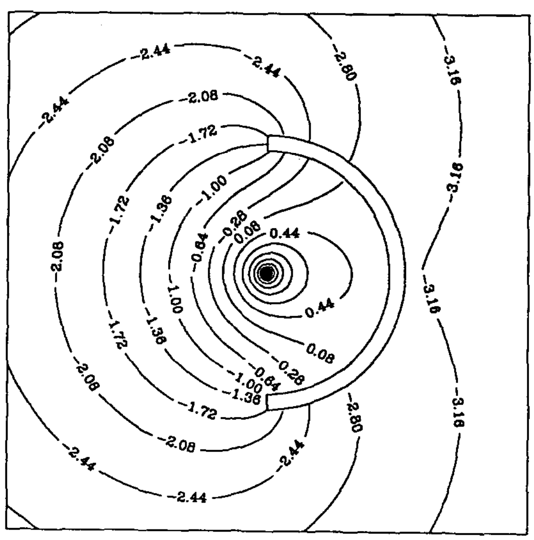

FIG. 7. Isothermal contours for $k_1/k_2 = 1$, $\alpha = 90°$.

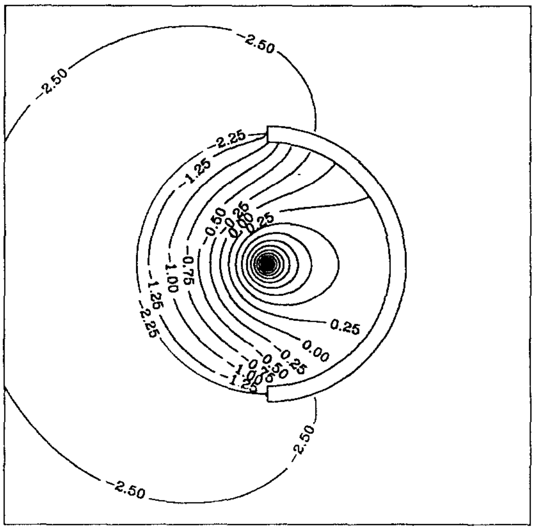

FIG. 8. Isothermal contours for $k_1/k_2 = 0.1$, $\alpha = 90°$.

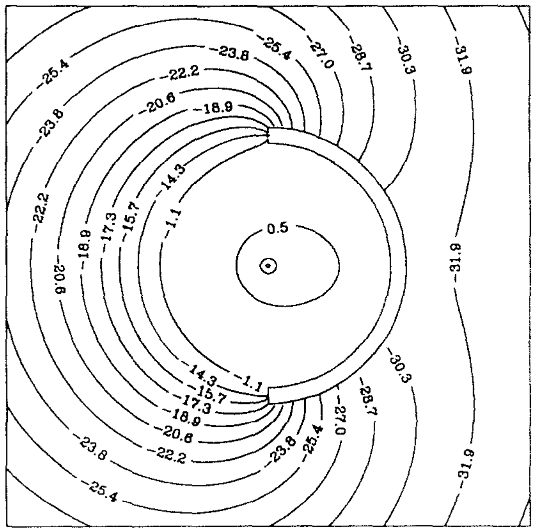

FIG. 9. Isothermal contours for $k_1/k_2 = 10$, $\alpha = 90°$.

for the case $k_1/k_2 = 10$. As the crack extends much further along the curved bond, say $\alpha = 90°$, the temperature contours for different material combinations $k_1/k_2 = 1$, 0.1, and 10 are displayed in Figs. 7–9, respectively. It is seen that the heat flow which travels in the direction of the gradient of the constant temperature contours tends to change its direction around tips of the crack. In this case, the presence of an insulated crack may play a more influential role on the temperature contours. Consequently, it affects the intensity of the temperature gradient over a significant portion of the interface.

HEAT FLUX INTENSITY FACTOR

The heat flux intensity factor, defined in equation (72), is introduced as a measure of the thermal energy intensification in the vicinity of the crack tip. Due to the symmetric property, only one of the crack tips needs to display the factor H for the crack length ranging from 0° to 180°. Referring to equation (74), the heat flux intensity factor is found to be dependent on the strength of a point heat source as well as the crack dimension. As the crack length is sufficiently small, the factor H is linearly proportional to the square-root of the crack length, and is very similar to the result for the corresponding problem with uniform heat flux applied at infinity [5]. Note that the solutions of heat flux in the given problem are independent of the material properties. This indicates that the amount of the thermal energy cumulated in the vicinity of the crack tip would not be influenced by the discontinuous jumps of the thermal properties across the interface. Namely, the heat flux intensity factor is independent

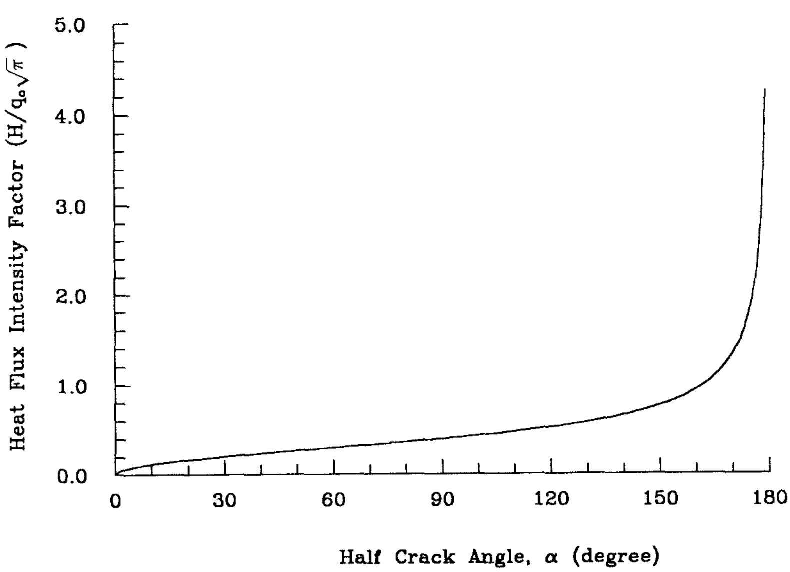

FIG. 10. Heat flux intensity factor at Tip-a (or Tip-b) vs half crack angle α.

of the heat conductivity coefficients. The dimensionless heat flux intensity factor $H/q_0\sqrt{\pi}$ vs the crack angle α is displayed in Fig. 10. It is seen that the factor $H/q_0\sqrt{\pi}$ increases smoothly from $\alpha = 0°$ to $150°$ and jumps to infinity near $\alpha = 180°$. This is because all the heat flow moves towards a relatively small portion of the interface and results in increasing the thermal energy intensification.

CONCLUSION

The general solutions of the steady-state heat conduction problem of a curvilinear crack in bonded dissimilar materials due to a point heat source have been obtained by using the Hilbert problem formulation and a special technique of analytic continuation. The heat fluxes or temperature gradients near tips of the curved crack are found to present the $1/\sqrt{\rho}$ singularity which is independent of the thermal properties of the materials. With this singular behavior, the heat flux intensity factor is then introduced to quantify the accumulation of thermal energy at the crack tips. Some numerical examples are given to illustrate the thermal behavior in the present analysis. It is indicated that the temperature or temperature gradients in each medium are dependent on its own material properties and will not affect each other. Moreover, the heat flux intensity factor is only dependent on the crack dimension for the given strength of a point heat source. In general, the system having a small crack would make the heat flux intensity factor smaller.

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